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Multiplicity of characteristics with Lagrangian boundary values on symmetric star-shaped hypersurfaces

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ABSTRACT

In this paper, the multiplicity of Lagrangian orbits on C^2 smooth compact symmetric star-shaped hypersurfaces with respect to the origin in $\mathbf{R}^{2n}$ is studied. These Lagrangian orbits begin from one Lagrangian subspace and end on another. An infinitely many existence result is proved via $\mathbf{Z}_2$-index theory. This is a multiplicity result about the Arnold Chord Conjecture in some sense, and is a generalization of the problem about the multiplicity of Lagrangian orbits beginning from and ending on the same Lagrangian subspace which was considered in the authors' previous paper [F. Guo, C. Liu, Multiplicity of Lagrangian orbits on symmetric star-shaped hypersurfaces, Nonlinear Anal. 69 (4) (2008) 1425–1436].

1. Introduction and main result

A C^2 hypersurface Σ in $\mathbf{R}^{2n}$ is called a star-shaped hypersurface, if it bounds an open set $\Gamma(\Sigma)$, and there exists a point $x_0(\Sigma) \in \Gamma(\Sigma)$ such that the tangent plane of Σ at any point $x \in \Sigma$ does not pass though $x_0(\Sigma)$. In this paper, we fix the point $x_0(\Sigma) = 0$ (the origin), and say that Σ is a star-shaped hypersurface with respect to the origin. In addition, in this paper, we suppose that Σ is symmetric with its center at the origin. We call this kind of hypersurfaces *the symmetric star-shaped hypersurfaces* in short.

For $z \in \Sigma$, let $N_\Sigma(z)$ be the unit outward normal vector of Σ at z. We consider the problem of finding $\tau > 0$ and an absolutely continuous curve $z : [0, \tau] \to \Sigma$ such that

$$\begin{cases} \dot{z}(t) = JN_\Sigma\big(z(t)\big), & \forall t \in [0, \tau], \\ z(0) \in L_1, & z(\tau) \in L_2, \end{cases} \tag{1.1}$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$ is the $2n \times 2n$ standard symplectic matrix with I_n being the $n \times n$ identity matrix, L_1 and L_2 are given Lagrangian subspaces in symplectic vector space $(\mathbf{R}^{2n}, \omega_0)$ with $\omega_0 = \sum_{i=1}^n dx_i \wedge dy_i$. A Lagrangian subspace L of $(\mathbf{R}^{2n}, \omega_0)$ is an n-dimensional subspace of $\mathbf{R}^{2n}$ satisfying $\omega_0|_L = 0$. Note that $L_i \cap \Sigma \neq \emptyset$. We denote by (τ, z) the (L_1, L_2)-*Lagrangian orbit* on Σ, which solves the problem (1.1).

If we choose a function $H_\Sigma \in C^1(\mathbf{R}^{2n}, \mathbf{R})$ satisfying $H_\Sigma^{-1}(1) = \Sigma$ with $\nabla H_\Sigma(z) \neq 0$ for all $z \in \Sigma$, then we can transform the problem (1.1) into the following nonlinear Hamiltonian system with fixed energy

$$\begin{cases} \dot{z}(t) = J\nabla H_\Sigma\big(z(t)\big), \\ H_\Sigma\big(z(t)\big) = 1, \quad t \in [0, \tau], \\ z(0) \in L_1, \qquad z(\tau) \in L_2. \end{cases} \tag{1.2}$$

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Similarly to the problem of periodic orbits, the existence and multiplicity of the solutions of the problem (1.1) are independent of the choice of the function H such that $\Sigma = H^{-1}(1)$. Moreover, we can transform the problem of finding the solutions of (1.2) to that of finding the critical points of a functional via variational principle.

In paper [15], the second author of this paper transformed the problem of Lagrangian intersections into a Hamiltonian system similar to the problem (1.1). From the viewpoint of the contact geometry, any compact star-shaped hypersurface Σ with respect to the origin in $\mathbf{R}^{2n}$ is a closed contact manifold with a contact form induced from the symplectic form ω_0, and $\tilde{L} = \Sigma \cap L$ is a Legendrian submanifold of Σ for any Lagrangian subspace L. So the existence of the problem (1.1) with $L_1 = L_2$ is a special case of the Arnold Chord Conjecture, see pp. 15–16 in paper [3], the conjecture said that on a closed contact manifold (precisely, S^{2n-1} with a standard contact structure ξ_0), for any closed Legendrian submanifold, there always exists a Reeb chord intersecting the Legendrian submanifold at least twice for any choice of contact form (see [19] for a proof of the conjecture). It is well known that one can transform this problem in the case of (S^{2n-1}, ξ_0) to the existence problem of (1.1) with Σ being a star-shaped hypersurface and L being a Lagrangian submanifold satisfying $\tilde{L} = \Sigma \cap L \neq \emptyset$ (cf. [11,12,18]). Any star-shaped hypersurface in $\mathbf{R}^{2n}$ is a contact manifold. The intersection of star-shaped hypersurface with respect to the origin with a Lagrangian subspace in $\mathbf{R}^{2n}$ is a Legendrian submanifold. The problem (1.1) is a generalization of the following problem

$$\begin{cases} \dot{z}(t) = JN_\Sigma\big(z(t)\big), & \forall t \in [0, \tau], \\ z(0), z(\tau) \in L, \end{cases} \tag{1.3}$$

where L is a given Lagrangian subspace in $(\mathbf{R}^{2n}, \omega_0)$.

The Maslov-type index theory corresponding to problem (1.3) was studied by the second author in paper [16], and for the special case $L = \{0\} \times \mathbf{R}^n$ it was studied in [17] via different methods. The existence and multiplicity of Lagrangian boundary solutions of asymptotically linear Hamiltonian systems were studied by using this Maslov-type index theory in [16]. The problem (1.3) is related with the Bolza problem (see [8], for example) and studied by the authors in paper [11] and [12]. In a recent paper [1], A. Abbondandolo and A. Figalli studied the Tonelli Lagrangians systems and Hamiltonians systems with various boundary value conditions.

In this paper, we study the multiplicity of the problem (1.1). We will give a proof of the following result.

Theorem 1.1. *If Σ is an arbitrary C^2 smooth compact symmetric star-shaped hypersurface with respect to the origin, then for arbitrary two Lagrangian subspaces L_1 and L_2 in $\mathbf{R}^{2n}$, Σ possesses infinitely many (L_1, L_2)-Lagrangian orbits.*

Suppose φ_H^t is the Hamiltonian flow of the Hamiltonian system $\dot{z}(t) = J\nabla H_\Sigma(z(t))$. We know that its restriction to the hypersurface Σ is a contact flow. We call the restriction of φ_H^t to Σ the *contact Hamiltonian flow* of Σ. We denote by $\tilde{L}_i = \Sigma \cap L_i$, $i = 1, 2$, the two Legendrian submanifold of the contact manifold Σ. The subset $\varphi_\Sigma(\tilde{L}_1) := \bigcup_{t>0} \varphi_H^t(\tilde{L}_1)$ of Σ is independent of the choice of H (in fact, it is an n-dimensional immersion submanifold of Σ). From Theorem 1.1, we have the following consequence about the intersection numbers.

Corollary 1.2. *For arbitrary two Lagrangian subspaces L_1 and L_2 in $\mathbf{R}^{2n}$, if Σ is a C^2 smooth compact symmetric star-shaped hypersurface with respect to the origin, then*

$$\sharp\big\{\tilde{L}_2 \cap \varphi_\Sigma(\tilde{L}_1)\big\} = \infty.$$

The result of Corollary 1.2 in some sense is related with the Lagrangian intersections. About this topic, one can refer to papers [5,7,9,10,13–15,20,21], etc.

Remark 1.3. We only need to prove Theorem 1.1 for the special case $L_1 = L_0$, where $L_0 = \{0\} \oplus \mathbf{R}^n$ is the standard Lagrangian subspace in $(\mathbf{R}^{2n}, \omega_0)$. It is well known that any Lagrangian subspace L can be transformed to L_0 by an orthogonal symplectic transformation. That is, there is an orthogonal symplectic matrix Q such that $QL = L_0$. Any orthogonal transformation transforms a symmetric star-shaped hypersurface to another one with the same properties.

Inspired by the result of Theorem 1.1, we have the following conjecture.

Conjecture. *For arbitrary two Lagrangian subspaces L_1 and L_2 in $(\mathbf{R}^{2n}, \omega_0)$, every star-shaped hypersurface Σ with respect to the origin possesses infinitely many (L_1, L_2)-Lagrangian orbits, i.e.*

$$\sharp\big\{\tilde{L}_2 \cap \varphi_\Sigma(\tilde{L}_1)\big\} = \infty.$$

Up to the authors' knowledge, the result

$$\sharp\big\{\tilde{L}_2 \cap \varphi_\Sigma(\tilde{L}_1)\big\} \geqslant 2$$

has not been proved for an arbitrary non-symmetric star-shaped hypersurface with respect to the origin.

2. Reduction to a Hamiltonian system

From now on, we suppose that Σ is a C^2 smooth symmetric star-shaped hypersurface in $\mathbf{R}^{2n}$ with respect to the origin, and denote by $\mathcal{C}$ the region around by Σ. Define the gauge function $j_\Sigma : \mathbf{R}^{2n} \to \mathbf{R}$ by $j_\Sigma(z) = \min\{\lambda > 0 \mid \frac{z}{\lambda} \in \mathcal{C}\}$, $\forall z \neq 0$ and $j_\Sigma(0) = 0$, then $j_\Sigma \in C(\mathbf{R}^{2n}, \mathbf{R}_+)$ with $\mathbf{R}_+ := [0, +\infty)$. Some properties of the function j_Σ can be found on p. 69 in the book [8]. Define $H_2(z) = j_\Sigma^2(z)$ then $H_2 \in C^1(\mathbf{R}^{2n}, \mathbf{R}_+) \cap C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}_+)$ is homogenous of degree two.

A function $\phi \in C^2(\mathbf{R}_+, \mathbf{R}_+)$ is said to be *admissible*, if it satisfies the following conditions:

(i) $\phi(0) = 0$, $\phi'(+\infty) := \lim_{t\to+\infty} \frac{\phi(t)}{t} > 0$,
(ii) $\phi''(t) < 0$, $\forall t \geqslant 0$,
(iii) $\sup_{t\geqslant 0} |\phi''(t)t| < +\infty$.

Such functions do exist, for example, $\phi(t) = t + \ln(t+1)$, $\forall t \geqslant 0$. If the function ϕ is admissible, from above three conditions we know that

$$\phi' \text{ is strictly decrease,} \quad \text{so} \quad 0 < \phi'(+\infty) < \phi'(t) \leqslant \phi'(0), \quad \forall t \geqslant 0, \tag{2.1}$$

and

$$\int_0^t \tau\phi''(\tau)\,d\tau < 0, \quad \forall t > 0, \quad \text{so} \quad \phi'(t)t - \phi(t) < 0, \quad \forall t > 0. \tag{2.2}$$

Choose an admissible function ϕ, which will be precisely determined later, define

$$H(z) = \phi\big(H_2(z)\big), \quad \forall z \in \mathbf{R}^{2n}.$$

Because Σ is symmetric with its center at the origin, we have that $H_2(z) = H_2(-z)$, $\forall z \in \mathbf{R}^{2n}$. Thus we get an even function $H \in C^1(\mathbf{R}^{2n}, \mathbf{R}_+) \cap C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R}_+)$, that is,

$$H(z) = H(-z), \quad \forall z \in \mathbf{R}^{2n}.$$

If we normalize the outward normal vector field $N_\Sigma(z)$ by requiring $(N_\Sigma(z), z) = 2\phi'(1)$ for every $z \in \Sigma$, then

$$\nabla H(z) = N_\Sigma(z), \quad \forall z \in \Sigma.$$

So we reduce the problem (1.1) for $L_1 = L_0$, $L_2 = L$ to the following fixed energy problem of Hamiltonian system with (L_0, L)-boundary value condition

$$\begin{cases} \dot z(t) = J\nabla H(z) = J\phi'\big(H_2(z)\big)\nabla H_2(z), \\ H\big(z(t)\big) = \phi(1) > 0, \quad t \in [0, \tau], \\ z(0) \in L_0, \qquad z(\tau) \in L, \end{cases} \tag{2.3}$$

where L is a Lagrangian subspace in $(\mathbf{R}^{2n}, \omega_0)$.

We first study the following fixed time problem

$$\begin{cases} \dot z(t) = J\nabla H(z), \quad t \in [0, 1], \\ z(0) \in L_0, \qquad z(1) \in L. \end{cases} \tag{2.4}$$

It is well known that for every Lagrangian subspace L in $(\mathbf{R}^{2n}, \omega_0)$, there exists a matrix $P \in Sp(2n) \cap O(2n) = U(n)$ such that $L = PL_0$. Because $U(n)$ is a Lie group and $P \in U(n)$, there exists a $2n \times 2n$ matrix M with $M^T + M = 0$ such that $P = e^M$. Moreover, the anti-symmetric matrix M satisfies the following conditions

$$JM = MJ, \tag{2.5}$$

$$(JM)^T = JM, \tag{2.6}$$

and so we have $e^M J = Je^M$.

Note that $P = e^M = e^{M+2k\pi J}$, so all the differences between the eigenvalues of the symmetric matrix JM and those of the symmetric matrix $J(M + 2k\pi J)$ are $2k\pi$. From now on, we choose the anti-symmetric matrix M in such way that the minimal eigenvalue of the symmetric matrix JM is larger than zero.

Set

$$\hat H(t, u) = H\big(e^{tM}u\big) + \frac{1}{2}(JMu, u).$$

If $z(t)$ is the solution of (2.4), let

$$u(t) = e^{tM^T} z(t), \tag{2.7}$$

then using (2.6), we get

$$\nabla\hat{H}(t,u)=e^{tM^T}\nabla H\big(e^{tM}u\big)+JMu. \tag{2.8}$$

From now on, $\nabla=\nabla_u$ is the gradient with respect to u. Using (2.8) and the fact that $M^T+M=0$, we have

$$\dot{u}(t)=e^{tM^T}\dot{z}(t)+M^Te^{tM^T}z(t)=e^{tM^T}J\nabla H\big(e^{tM}u(t)\big)+M^Tu(t)=J\nabla\hat{H}\big(t,u(t)\big), \tag{2.9}$$

and

$$u(0)=z(0)\in L_0,\qquad u(1)=e^{M^T}z(1)\in P^TL=L_0. \tag{2.10}$$

From (2.9) and (2.10), we know that under the transform (2.7), one solution of (2.4) is corresponding to that of the following problem

$$\begin{cases}\dot{u}(t)=J\nabla\hat{H}(t,u), & t\in[0,1],\\ u(0),u(1)\in L_0.\end{cases} \tag{2.11}$$

On the other hand, if $u(t)$ is the solution of (2.11), let

$$z(t)=e^{tM}u(t),$$

then $z(t)$ is the solution (2.4) and $\tilde{z}(t):=\rho^{-1/2}z(\frac{\phi'(1)t}{\phi'(\rho)})$ is the solution of the system (2.3) with $\tau=\frac{\phi'(\rho)}{\phi'(1)}$, where $\rho=H_2(z)$.

By direct computation, $H''(z)=\phi''(H_2(z))\nabla H_2(z)\nabla H_2^T(z)+\phi'(H_2(z))H_2''(z)$. In view of H_2 is homogenous of degree two and ϕ is admissible, H'' is bounded in $\mathbf{R}^{2n}\setminus\{0\}$ by the admissible condition (iii). From the fact that

$$\hat{H}''(t,u)=e^{tM^T}H''\big(e^{tM}u\big)e^{tM}+JM,$$

we know that $\hat{H}''(t,u)$ is also bounded in $[0,1]\times\{\mathbf{R}^{2n}\setminus\{0\}\}$. So we can choose a constant $G>0$ such that

$$\big|\hat{H}''(t,u)\big|\leqslant G,\quad \forall(t,u)\in[0,1]\times\big\{\mathbf{R}^{2n}\setminus\{0\}\big\}. \tag{2.12}$$

Define

$$\hat{H}_K(t,u)=\hat{H}(t,u)+\frac{K}{2}|u|^2,$$

then we can choose $K>G$ such that $\hat{H}_K\in C^1([0,1]\times\mathbf{R}^{2n},\mathbf{R}_+)\cap C^2([0,1]\times\{\mathbf{R}^{2n}\setminus\{0\}\},\mathbf{R}_+)$ is strictly convex in the sense that

$$\big(\nabla\hat{H}_K(t,u_1)-\nabla\hat{H}_K(t,u_2),u_1-u_2\big)\geqslant(K-G)|u_1-u_2|^2>0,\quad \forall u_1\neq u_2\in\mathbf{R}^{2n}\setminus\{0\}.$$

The *Fenchel dual* of $\hat{H}_K$ is defined by

$$\hat{H}_K^*\big(t,z^*\big)=\sup_{u\in R^{2n}}\big\{\big(z^*,u\big)-\hat{H}_K(t,u)\big\},\quad \forall z^*\in\mathbf{R}^{2n},\ t\in[0,1],$$

then $\hat{H}_K^*\in C^1([0,1]\times\mathbf{R}^{2n},\mathbf{R}_+)\cap C^2([0,1]\times\{\mathbf{R}^{2n}\setminus\{0\}\},\mathbf{R}_+)$ is also strictly convex, moreover, $\hat{H}_K^*$ has the following properties (see p. 85, Proposition 15 of [8]):

$$\nabla\hat{H}_K^*\big(t,z^*\big)=u\quad\text{if and only if}\quad z^*=\nabla\hat{H}_K(t,u), \tag{2.13}$$

$$\hat{H}_K^*\big(t,z^*\big)=\big(u,z^*\big)-\hat{H}_K(t,u)\quad\text{if and only if}\quad z^*=\nabla\hat{H}_K(t,u). \tag{2.14}$$

From now on, fix the positive number $K\notin\pi\mathbf{Z}$ and define two spaces as following

$$\mathcal{W}:=\big\{z\in W^{1,2}\big([0,1],\mathbf{R}^{2n}\big)\ \big|\ z(0),z(1)\in L_0\big\},\qquad \mathcal{L}:=L^2\big([0,1],\mathbf{R}^{2n}\big).$$

These two spaces are Hilbert spaces with inner products defined respectively by $\langle z_1,z_2\rangle_{\mathcal{W}}=\int_0^1\{(z_1,z_2)+(\dot{z}_1,\dot{z}_2)\}\,dt$ and $\langle z_1,z_2\rangle_{\mathcal{L}}=\int_0^1(z_1,z_2)\,dt$. From now on, we denote the inner product and norm in $\mathcal{L}$ by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ respectively.

The operator $-J\frac{d}{dt}:\mathcal{W}\to\mathcal{L}$ is self-adjoint in the inner product of $\mathcal{L}$, and its spectrums are $\{k\pi\mid k\in\mathbf{Z}\}$ with all spectrums being eigenvalues of the operator $-J\frac{d}{dt}$, the corresponding eigen-subspaces are

$$E_k:=\operatorname{span}\left\{e_{kj}(t)=\begin{pmatrix}-\sin(k\pi t)e_j\\ \cos(k\pi t)e_j\end{pmatrix}:\ j=1,\ldots,n\right\},$$

where e_j, $j=1,\ldots,n$, is the standard basis of $\mathbf{R}^n$. Obviously, $\dim E_k=n$ and $\{e_{kj}\mid k\in\mathbf{Z},\ \ j=1,\ldots,n\}$ is the standard orthogonal basis of $\mathcal{W}$ in the inner product of $\mathcal{L}$. So every $u\in\mathcal{W}$ can be written as

$$u(t) = \sum_{k\in\mathbf{Z}} \sum_{j=1}^{n} u_{kj} e_{kj}(t)$$

with $\sum_{k\in\mathbf{Z}} \sum_{j=1}^{n} (1 + k^2\pi^2) u_{kj}^2 < +\infty$. Through direct computation, we have

$$\int_0^1 |-J\dot{u} + Ku|^2 \, dt = \sum_{k\in\mathbf{Z}} \sum_{j=1}^{n} (K + k\pi)^2 u_{kj}^2, \tag{2.15}$$

$$\int_0^1 (-J\dot{u} + Ku, u) \, dt = \sum_{k\in\mathbf{Z}} \sum_{j=1}^{n} (K + k\pi) u_{kj}^2. \tag{2.16}$$

Define a dual functional $F_K : \mathcal{W} \to \mathbf{R}$ by

$$F_K(u) = \int_0^1 \left[\hat{H}_K^*\big(t, -J\dot{u}(t) + Ku(t)\big) - \frac{1}{2}\big(-J\dot{u}(t) + Ku(t), u(t)\big) \right] dt, \quad \forall u \in \mathcal{W}.$$

From the fact that $H(-z) = H(z)$ and the definition of Fenchel dual, we know that $\hat{H}_K^*(t, -u) = \hat{H}_K^*(t, u)$, so F_K is an even functional on $\mathcal{W}$.

Proposition 2.1. *For above K with $K \notin \pi\mathbf{Z}$, u is a critical point of F_K if and only if u is C^1 and u is a solution of* (2.11)*.*

Proof. By direct computation,

$$\langle F_K'(u), h \rangle_{\mathcal{W}} = \int_0^1 \big[\big(\nabla \hat{H}_K^*(t, -J\dot{u} + Ku) - u, -J\dot{h} + Kh\big)\big] \, dt, \quad \forall h \in \mathcal{W}. \tag{2.17}$$

If u is a solution of (2.11), then $-J\dot{u} + Ku = \nabla \hat{H}_K(t, u)$, using (2.13), we have $\nabla \hat{H}_K^*(t, -J\dot{u} + Ku) = u$, so $\langle F_K'(u), h \rangle_{\mathcal{W}} = 0$, $\forall h \in \mathcal{W}$.

Suppose u satisfies that $\langle F_K'(u), h \rangle_{\mathcal{W}} = 0$, $\forall h \in \mathcal{W}$. From the condition $K \notin \pi\mathbf{Z}$ we know that the operator $-J\frac{d}{dt} + Kid : \mathcal{W} \to \mathcal{L}$ is invertible, so there exists $h \in \mathcal{W}$ such that $\nabla \hat{H}_K^*(t, -J\dot{u} + Ku) - u = -J\dot{h} + Kh$. For this $h \in \mathcal{W}$, we have $\int_0^1 |\nabla \hat{H}_K^*(t, -J\dot{u} + Ku) - u|^2 \, dt = 0$, then we get $\nabla \hat{H}_K^*(t, -J\dot{u} + Ku) = u$, a.e. on $[0, 1]$, using (2.13), we get $-J\dot{u} + Ku = \nabla \hat{H}_K(t, u)$, a.e. on $[0, 1]$, i.e.

$$\dot{u} = J\nabla \hat{H}(t, u), \quad \text{a.e. on } [0, 1]. \tag{2.18}$$

By the Sobolev imbedding theorem, we have $u \in C([0, 1], \mathbf{R}^{2n})$. Since the fact that u satisfies that (2.18), the derivation $\dot{u}$ can be extended continuously to all t on $[0, 1]$. This proves $u \in C^1([0, 1], \mathbf{R}^{2n})$, i.e., $u(t)$ is the solution of (2.11). □

Proposition 2.2. *$u \equiv 0$ is the unique trivial critical point of F_K and $F_K(0) = 0$. If u is a nontrivial critical point, then$F_K(u) < 0$.*

Proof. From the definition of $\hat{H}$, we know that

$$\big(\nabla \hat{H}(t, u), u\big) = \phi'\big(H_2\big(e^{tM}u\big)\big)\big(\nabla H_2\big(e^{tM}u\big), e^{tM}u\big) + (JMu, u) > 0, \quad \forall u \neq 0. \tag{2.19}$$

Note that the hypersurface is star-shaped, so the first term is larger than zero. In addition, as we choose the matrix M such that the minimal eigenvalue of the matrix JM is larger than zero, the second term is also larger than zero. From (2.8) and (2.19), we have

$$\nabla \hat{H}(t, 0) = 0, \qquad \nabla \hat{H}(t, u) \neq 0, \quad \forall u \neq 0. \tag{2.20}$$

So we know that the system (2.11) has one unique trivial solution $u \equiv 0$.

If u is a nontrivial critical point, from Proposition 2.1, we know that $\dot{u} = J\nabla \hat{H}(t, u)$, so $-J\dot{u} + Ku = \nabla \hat{H}_K(t, u)$, using (2.14) we have $\hat{H}_K^*(t, -J\dot{u} + Ku) = (-J\dot{u} + Ku, u) - \hat{H}_K(t, u)$. Now we can estimate the critical value $F_K(u)$ by using (2.2):

$$
\begin{aligned}
F_K(u) &= \int_0^1 \left[\hat{H}_K^*(t, -J\dot{u} + Ku) - \frac{1}{2}(-J\dot{u} + Ku, u)\right] dt \\
&= \int_0^1 \left[\frac{1}{2}(-J\dot{u} + Ku, u) - \hat{H}_K(t, u)\right] dt \\
&= \int_0^1 \left[\frac{1}{2}\big(\nabla \hat{H}(t, u), u\big) - \hat{H}(t, u)\right] dt \\
&= \int_0^1 \left[\frac{1}{2}\phi'\big(H_2\big(e^{tM}u\big)\big)\big(\nabla H_2\big(e^{tM}u\big), e^{tM}u\big) - \phi\big(H_2\big(e^{tM}u\big)\big)\right] dt \\
&= \int_0^1 \left[\phi'\big(H_2\big(e^{tM}u\big)\big)H_2\big(e^{tM}u\big) - \phi\big(H_2\big(e^{tM}u\big)\big)\right] dt < 0.
\end{aligned}
$$

We complete the proof. $\square$

Thus we know that under the transform (2.7), a solution of (2.3) is corresponding exactly to one solution of (2.11), and one can reduce the solutions of the system (2.11) to the critical points of the functional F_K on $\mathcal{W}$.

3. Proof of Theorem 1.1 via $\mathbf{Z}_2$-index theory

Now, we recall the definition and some properties of $\mathbf{Z}_2$-index (or Krasnoselskii genus, see paper [2] or book [6,22]). Suppose that E is a Banach space and define the following set

$$\mathcal{E} := \{A \subset E \mid A \text{ is closed in } E \text{ and symmetric with its center at the origin}\}.$$

For every close set $A \in \mathcal{E}$, we define the $\mathbf{Z}_2$-index of A by

$$\gamma(A) = \begin{cases} \min\{m \in \mathbf{N} \mid \exists \varphi : A \to \mathbf{R}^m \setminus \{0\} \text{ odd and continuous}\}, & \text{if } A \neq \emptyset, \\ 0, & \text{if } A = \emptyset, \\ +\infty, \quad \text{if there is no } \varphi : A \to \mathbf{R}^m \setminus \{0\} \text{ odd and continuous, } \forall m = 1, 2, \ldots. \end{cases}$$

The $\mathbf{Z}_2$-index of a closed set A has the following properties (see [2,22], for example).

Proposition 3.1. (1) $\gamma(A) = 0$ *if and only if* $A = \emptyset$.

(2) $\gamma(S^n) = n + 1$, *where* S^n *is the sphere in* $\mathbf{R}^{n+1}$.

(3) *If* $\gamma(A) \geqslant 2$, *then* A *possesses infinitely many points* (*see p. 96, Observation* 5.5 *of* [22]).

(4) *Let* V *be a* k*-dimensional subspace of* E, $V^\perp$ *is an algebraically or topologically complementary subspace, if* $\gamma(A) > k$, *then* $A \cap V^\perp \neq \emptyset$.

(5) *If* A *is compact, then* $\gamma(A) < +\infty$.

Lemma 3.2. *(See [6,22].) Suppose that* E *is a Banach space,* f *is an even functional and satisfies* (*PS*) *condition, that is, for a sequence* $\{z_j\} \subset E$ *with* $f(z_j)$ *bounded and* $f'(z_j) \to 0$ *in* E *as* $j \to +\infty$, *then there exists a subsequence converging to* z *in* E. *Define*

$$c_k = \inf_{\gamma(A) \geqslant k} \sup_{x \in A \in \mathcal{E}} f(x), \quad k = 1, 2, \ldots,$$

then we have that

(1) *if* $-\infty < c_k < +\infty$, *then* c_k *is the critical value of* f,
(2) *if* $-\infty < c := c_k = c_{k+1} = \cdots = c_{k+m-1} < +\infty$, *then* $\gamma(\mathcal{K}_c) \geqslant m$, *where* $\mathcal{K}_c := \{x \in E \mid f'(x) = 0 \text{ and } f(x) = c\}$,
(3) $c_k \leqslant c_{k+1}$.

Lemma 3.3. *Suppose that* E *is a Banach space, and* f *is an even functional satisfying the* (*PS*) *condition and* $f(0) = 0$, *then*

(1) *if there exists an* m*-dimensional subspace* V_1 *of* E *and a sphere* $S_\rho(0)$ *in* V_1 *such that* $\sup_{x \in V_1 \cap S_\rho} f(x) < 0$, *then* $c_k < 0$ *for* $1 \leqslant k \leqslant m$,
(2) *if there exists a* p*-dimensional subspace* V_2 *such that* $\inf_{x \in V_2^\perp} f(x) > -\infty$, *then* $c_k > -\infty$ *for* $k > p$,

(3) *if $m > p$ in (1) and (2) above, then f has at least $m-p$ pairs distinct critical points $x_{p+1}, x_{p+2}, \dots, x_m, -x_{p+1}, -x_{p+2}, \dots, -x_m$, which correspond to the critical value $c_{p+1} \leqslant c_{p+2} \leqslant \dots \leqslant c_m < 0$ defined by*

$$c_k = \inf_{\gamma(A)\geqslant k} \sup_{x\in A\in\mathcal{E}} f(x), \quad k = p+1, p+2, \dots, m.$$

Proof. One can find a proof of the lemma in our previous paper [12]. For the readers' convenience, we prove it here.

Step 1. From the definition of c_m and the fact that $\gamma(V_1 \cap S_\rho) = m$, we have

$$c_m := \inf_{\gamma(A)\geqslant m} \sup_{x\in A\in\mathcal{E}} f(x) \leqslant \sup_{x\in V_1\cap S_\rho} f(x) < 0.$$

So $c_k \leqslant c_m < 0$ for $k \leqslant m$.

Step 2. From the result (4) in Proposition 3.1, we know that if $\gamma(A) > p$, then $A \cap V_2^\perp \neq \emptyset$. From the definition of c_{p+1} we know that

$$c_{p+1} := \inf_{\gamma(A)\geqslant p+1} \sup_{x\in A} f(x) \geqslant \inf_{\gamma(A)\geqslant p+1} \sup_{x\in A\cap V_2^\perp} f(x) \geqslant \inf_{x\in V_2^\perp} f(x) > -\infty.$$

So $c_k \geqslant c_{p+1} > -\infty$ for $k \geqslant p+1$.

Step 3. By Steps 1, 2 and Lemma 3.2, we know that $c_k, k = p+1, p+2, \dots, m$ defined by $c_k := \inf_{\gamma(A)\geqslant k} \sup_{x\in A} f(x)$ are the critical values of f. If $c_{p+1} < c_{p+2} < \dots < c_m$, then we have at least $m-p$ pairwise distinct critical points

$$x_{p+1}, x_{p+2}, \dots, x_m, -x_{p+1}, -x_{p+2}, \dots, -x_m,$$

which correspond to $m-p$ distinct critical values. If there exist $i \neq j \in \{p+1, p+2, \dots, m\}$ such that $c_i = c_j$, then from Lemma 3.2 we know that $\gamma(\mathcal{K}_{c_i}) \geqslant 2$, where $\mathcal{K}_{c_i} := \{x \in E \mid f'(x) = 0 \text{ and } f(x) = c_i\}$, using Proposition 3.1, $\mathcal{K}_{c_i}$ possesses infinity many points. The proof is completed. □

Because Σ is compact in $\mathbf{R}^{2n}$, there exist two balls by which Σ is pinched, that is, if $\mathcal{C}$ denotes the region bounded by Σ, then there exist two balls $\mathcal{B} := \{z \in \mathbf{R}^{2n} \mid \frac{R}{2}|z|^2 \leqslant 1\}$ and $\beta\mathcal{B} := \{z \in \mathbf{R}^{2n} \mid \frac{R}{2}|z|^2 \leqslant \beta\}$ such that $\mathcal{B} \subset \mathcal{C} \subset \beta\mathcal{B}$ for some $\beta > 1$, so

$$\frac{R}{2\beta}|z|^2 \leqslant H_2(z) \leqslant \frac{R}{2}|z|^2, \quad \forall z \in R^{2n}. \tag{3.1}$$

Set $b_s = (\frac{1}{2R} - s)\pi$, $G_s(t,u) = b_s H_2(e^{tM}u) + \frac{1}{2}(JMu, u)$, then

$$\nabla G_s(t,u) = b_s e^{tM^T} \nabla H_2\big(e^{tM}u\big) + JMu. \tag{3.2}$$

If for every $s \in (0, \frac{1}{4R})$, the following system

$$\begin{cases} \dot{u}(t) = J\nabla G_s\big(t, u(t)\big), & t \in [0,1], \\ u(0), u(1) \in L_0 \end{cases} \tag{3.3}$$

always has a nontrivial solution $u_s(t)$, then $z_s(t) = e^{tM}u_s(t)$ is a solution of

$$\begin{cases} \dot{z}(t) = Jb_s\nabla H_2\big(z(t)\big), & t \in [0,1], \\ z(0) \in L_0, \quad z(1) \in L, \end{cases} \tag{3.4}$$

and $\tilde{z}_s(t) := \rho_s^{-1/2} z_s(\frac{\phi'(1)t}{b_s})$ is a solution of the fixed energy problem:

$$\begin{cases} \dot{z}(t) = J\nabla H\big(z(t)\big), \\ H\big(z(t)\big) = \phi(1), \quad t \in \left[0, \dfrac{b_s}{\phi'(1)}\right], \\ z(0) \in L_0, \quad z\left(\dfrac{b_s}{\phi'(1)}\right) \in L, \end{cases}$$

where $\rho_s = H_2(z_s(t)) > 0$. That is, for every $s \in (0, \frac{1}{4R})$, $(\frac{b_s}{\phi'(1)}, \tilde{z}_s)$ solves the problem (2.3). Since the infinitely different choices of b_s, we get infinite many solutions of (2.3). That is, Theorem 1.1 is proved in this case.

Now we consider our problem with the following condition

(C) *there exists an $s_0 \in (0, \frac{1}{4R})$ such that the system* (3.3) *has no nontrivial solution.*

For the s_0 determined by condition (C) and the number $r > 0$ determined later, we choose an admissible function ϕ by specifying $\phi'(0)$ and $\phi'(+\infty)$. Precisely, we choose $\phi'(0)$ and $\phi'(+\infty)$ such that

$$\phi'(0)=\pi\left(\frac{\beta}{R}+r\right),\qquad \phi'(+\infty)=\pi\left(\frac{1}{2R}-s_0\right),$$

such admissible functions ϕ do exist. For example, we set

$$\phi(t)=\pi\left(\frac{1}{2R}-s_0\right)t+\pi\left[\left(\frac{\beta}{R}+r\right)-\left(\frac{1}{2R}-s_0\right)\right]\ln(t+1),\quad t\geqslant 0.$$

In this section, we set

$$a=\pi\left(\frac{\beta}{R}+\frac{r}{2}\right)\in\big(0,\phi'(0)\big),\qquad b=\phi'(+\infty)=\pi\left(\frac{1}{2R}-s_0\right). \tag{3.5}$$

Lemma 3.4. *Suppose that the function ϕ satisfies $\phi'(+\infty)=\pi(\frac{1}{2R}-s_0)$, then the functional F_K satisfies the strong (PS) condition, i.e., for a sequence $\{u_j\}\subset\mathcal{W}$ satisfying $F'_K(u_j)\to 0$ in $\mathcal{W}$ as $j\to+\infty$, there exists a subsequence of $\{u_j\}$ converging to $u\in\mathcal{W}$ and $F'_K(u)=0$.*

Proof. We follow the ideas of [4] and [12]. By direct computation, we have

$$\langle F'_K(u),h\rangle_{\mathcal{W}}=\int_0^1\big[\big(\nabla\hat H^*_K(t,-J\dot u+Ku)-u,-J\dot h+Kh\big)\big]\,dt,\quad\forall h\in\mathcal{W}. \tag{3.6}$$

Suppose that $\{u_j\}\subset\mathcal{W}$ satisfying $F'_K(u_j)\to 0$, $j\to+\infty$ in $\mathcal{W}$, we will prove that there exists a subsequence of $\{u_j\}$ converging to some $u\in\mathcal{W}$ with $F'_K(u)=0$.

From (3.6) we know that $\Pi_K u_j-\Pi_K\nabla\hat H^*_K(t,-\Pi_K u_j):=\eta_j\to 0$ in $\mathcal{L}$, where $\Pi_K:\mathcal{W}\to\mathcal{L}$ is defined by $\Pi_K z:=J\dot z-Kz$, which has bounded inverse. So we have

$$u_j-\nabla\hat H^*_K(t,-\Pi_K u_j):=\epsilon_j\to 0,\quad j\to+\infty,\ \text{in }\mathcal{W}. \tag{3.7}$$

In view of (2.13), we have

$$-J\dot u_j+Ku_j=\nabla\hat H_K(t,u_j-\epsilon_j)=e^{tM^T}\phi'\big(H_2\big(e^{tM}(u_j-\epsilon_j)\big)\big)\nabla H_2\big(e^{tM}(u_j-\epsilon_j)\big)+JM(u_j-\epsilon_j)+K(u_j-\epsilon_j),$$

that is,

$$\phi'\big(H_2\big(e^{tM}(u_j-\epsilon_j)\big)\big)e^{tM^T}\nabla H_2\big(e^{tM}(u_j-\epsilon_j)\big)+JM(u_j-\epsilon_j)+J\dot u_j=K\epsilon_j,\quad\forall j. \tag{3.8}$$

Claim. *$\|u_j\|_{C^{0,\alpha}}\leqslant c$ holds for some constant $c>0$ and $\alpha\in(0,1/2)$.*

Otherwise, there is a subsequence, for example, $\|u_j\|_{C^{0,\alpha}}\to+\infty$ as $j\to+\infty$. Set $v_j=\frac{u_j}{\|u_j\|_{C^{0,\alpha}}}$, $w_j=v_j-\frac{\epsilon_j}{\|u_j\|_{C^{0,\alpha}}}$ for j large. Using (3.8) and the homogenous of ∇H_2, we have

$$\phi'\big(H_2\big(e^{tM}(u_j-\epsilon_j)\big)\big)e^{tM^T}\nabla H_2\big(e^{tM}w_j\big)+JMw_j+J\dot v_j=K\frac{\epsilon_j}{\|u_j\|_{C^{0,\alpha}}},\quad\forall j\text{ large}. \tag{3.9}$$

From (2.1) we know that $\phi'(H_2(e^{tM}(u_j-\epsilon_j)))\in[\phi'(0),\phi'(+\infty))$, $\forall t\in[0,1]$. From the definition of w_j we know that $|w_j|$ is bounded, then from (3.9) we get that $\|\dot v_j\|$ is bounded via the homogeneity of degree 1 of ∇H_2, and from the inequality $\|z-z(0)\|\leqslant\|\dot z\|$, $\forall z\in\mathcal{W}$, we also get $\|v_j\|_{\mathcal{W}}$ is also bounded. From the imbedding $\mathcal{W}\hookrightarrow C^{0,\alpha}$ is compact for $\alpha\in(0,1/2)$, there exists a subsequence of v_j, still denoted by v_j, strongly convergent to v in $C^{0,\alpha}$, $\|v\|_{C^{0,\alpha}}=1$, write $v_j\to v\in C^{0,\alpha}$, $j\to+\infty$. ($v_j\to v$ in $\mathcal{L}$ is also true, since we have the imbedding $C^{0,\alpha}\hookrightarrow\mathcal{L}$.) From the fact $\|\dot v_j\|$ is bounded and $\phi'(H_2(e^{tM}(u_j-\epsilon_j)))\in[\phi'(0),\phi'(+\infty)]$, we get two weakly convergent subsequence $\dot v_j\rightharpoonup\dot v$ in $\mathcal{L}$ and $\xi_j:=\phi'(H_2(e^{tM}(u_j-\epsilon_j)))\rightharpoonup\xi$ in $\mathcal{L}([0,1],\mathbf{R}_+)$. So from (3.9) we know that in the weak sense in $\mathcal{L}$, there holds

$$\xi(t)e^{tM^T}\nabla H_2\big(e^{tM}v(t)\big)+JMv(t)=-J\dot v(t),\quad v\neq 0,\ \xi(t)\in\big[\phi'(0),\phi'(+\infty)\big]. \tag{3.10}$$

Since we have known that $\nabla H_2(z)=0\Leftrightarrow z=0$ and the solution of (3.10) with certain initial condition is unique, we know that $|v(t)|\geqslant\theta>0$ for all $t\in[0,1]$, then $|v_j(t)|\geqslant\theta/2>0$ for all $t\in[0,1]$ and $j\geqslant N$ (N large enough), so we have that $|u_j(t)|=\|u_j\|_{C^{0,\alpha}}|v_j(t)|\to+\infty$ uniformly in $t\in[0,1]$, as $j\to+\infty$. From the fact that $\epsilon_j\to 0$ in $\mathcal{W}$ and the fact that $\mathcal{W}$ is imbedded in C^0, we know that $\epsilon_j\to 0$ uniformly on $[0,1]$ as $j\to\infty$, so we have $|\epsilon_j(t)|\leqslant 1/2$ for j large enough and $|u_j(t)-\epsilon_j(t)|\to+\infty$ on $[0,1]$. Furthermore, using the homogeneity of degree 1 of ∇H_2 again, we have $|\nabla H_2(e^{tM}(u_j-\epsilon_j))|\leqslant C'|u_j-\epsilon_j|$, then $|\nabla H_2(e^{tM}w_j)|\leqslant C'|v_j-\frac{\epsilon_j}{\|u_j\|_{C^{0,\alpha}}}|\leqslant C''(1+|v|)$ for j large enough. From the Lebesgue's dominated convergence theorem, we know that $\phi'(H_2(e^{tM}(u_j-\epsilon_j)))e^{tM^T}\nabla H_2(e^{tM}w_j)\to\phi'(+\infty)e^{tM^T}\nabla H_2(e^{tM}v)$ in $\mathcal{L}$. Let $j\to+\infty$ in (3.9) then we get

$$-J\dot{v}=\phi'(+\infty)e^{tM^T}\nabla H_2\big(e^{tM}v\big)+JMv,\quad \|v\|_{C^{0,\alpha}}=1,\ v\in\mathcal{W},\tag{3.11}$$

that is, we get a nontrivial solution of (3.3), which is a contradiction under the condition (C).

From the above Claim: $\|u_j\|_{C^{0,\alpha}}\leqslant c$ we know that there exists one subsequence of u_j, still denoted by u_j, such that $u_j\to u$ in $C^0([0,1],\mathbf{R}^{2n})$. From (3.8), we get

$$\begin{aligned}\dot{u}_j&=J\big[\phi'\big(H_2\big(e^{tM}(u_j-\epsilon_j)\big)\big)e^{tM^T}\nabla H_2\big(e^{tM}(u_j-\epsilon_j)\big)+JM(u_j-\epsilon_j)-K\epsilon_j\big]\\&\to J\big[\phi'\big(H_2\big(e^{tM}u\big)\big)e^{tM^T}\nabla H_2\big(e^{tM}u\big)+JMu\big],\quad\text{in }\mathcal{L}.\end{aligned}$$

So $u_j\to u$ in $\mathcal{W}$ and $\dot{u}=J[e^{tM^T}\phi'(H_2(e^{tM}u))\nabla H_2(e^{tM}u)+JMu]=J\nabla\hat{H}(t,u)$, that is $F_K'(u)=0$. $\square$

Proposition 3.5. *We can choose suitable ϕ such that there exist two subspaces V_1 and V_2 of $\mathcal{W}$ with the dimensions m and p respectively satisfying*

(1) $\sup_{x\in V_1\cap S_\rho}F_K(x)<0$, *where $S_\rho:=S_\rho(0)$ is a sphere in V_1 with the radius ρ,*
(2) $\inf_{x\in V_2^\perp}F_K(x)>-\infty$,
(3) $m-p=n(E(1+\frac{rR}{2\beta}+\frac{\lambda_{\min}}{\pi})-E(1-2Rs_0+\frac{\lambda_{\max}}{\pi}))$, *where $E(\chi)$ is defined by $E(\chi)=\min\{k\in\mathbf{Z}\mid k\geqslant\chi\}$, $\lambda_{\min}$ and $\lambda_{\max}$ are the minimal and maximal eigenvalue of the symmetric matrix JM respectively.*

Proof. The proof is divided into three steps according to Lemma 3.3.

Step 1. From the fact that $\lim_{t\to 0^+}\frac{\phi(t)}{t}=\phi'(0)>a$ we know that there exists small $\delta=\delta(a)$ such that

$$\phi(t)\geqslant at,\quad\text{for }0<t\leqslant\delta.\tag{3.12}$$

Using (3.12) and (3.1), we get

$$H(z)\geqslant aH_2(z)\geqslant\frac{aR}{2\beta}|z|^2,\quad\forall|z|\leqslant\delta.$$

From the definition of $\hat{H}_K(t,u)$ and the fact that e^{tM} is an orthogonal matrix, we have that

$$\hat{H}_K(t,u)\geqslant aH_2\big(e^{tM}u\big)+\frac{1}{2}(JMu,u)+\frac{K}{2}|u|^2\geqslant\frac{aR}{2\beta}\big|e^{tM}u\big|^2+\frac{1}{2}(\lambda_{\min}+K)|u|^2=\frac{1}{2}\left(\frac{aR}{\beta}+\lambda_{\min}+K\right)|u|^2,\quad\forall|u|\leqslant\delta.$$

From the definition of the Fenchel dual we have

$$\hat{H}_K^*(t,u)\leqslant\frac{1}{2(K+\frac{aR}{\beta}+\lambda_{\min})}|u|^2,\quad\forall|u|\leqslant\delta.\tag{3.13}$$

Define

$$B_K(u)=\frac{1}{2}\int_0^1\left[\left(K+\frac{aR}{\beta}+\lambda_{\min}\right)^{-1}|-J\dot{u}+Ku|^2-(-J\dot{u}+Ku,u)\right]dt,\quad\forall u\in\mathcal{W}.\tag{3.14}$$

Using (2.15) and (2.16), we have

$$B_K(u)=\frac{1}{2}\sum_{k\in\mathbf{Z}}\sum_{j=1}^{n}\left(K+\frac{aR}{\beta}+\lambda_{\min}\right)^{-1}(K+k\pi)\left(k\pi-\frac{aR}{\beta}-\lambda_{\min}\right)u_{kj}^2.\tag{3.15}$$

Define the subspace V_1 in $\mathcal{W}$ by

$$V_1=\operatorname{span}\left\{e_{kj}\;\middle|\;-\frac{K}{\pi}<k<\frac{1}{\pi}\left(\frac{aR}{\beta}+\lambda_{\min}\right),\ k\in\mathbf{Z},\ j=1,\dots,n\right\},$$

and its dimension is defined by $m:=n(E(\frac{K}{\pi})+E(\frac{aR}{\pi\beta}+\frac{\lambda_{\min}}{\pi})-1)$.

From (3.13)–(3.15) and the fact that all the norms of finite dimensional space are equivalent, we know that there exists small $\rho=\rho(\delta)$ such that

$$F_K|_{V_1\cap S_\rho}\leqslant B_K|_{V_1\cap S_\rho}<0.$$

Step 2. From the fact that $\lim_{t\to+\infty}\frac{\phi(t)}{t}=b<2b$ we know that there exists a large positive number N such that $\frac{\phi(t)}{t}<2b$ for $t>N$, and $\phi(t)$ is bounded for $t\in[0,N]$, so there exists a constant C such that

$$\phi(t)\leqslant 2bt+C,\quad\text{for all }t\geqslant 0.\tag{3.16}$$

Using (3.16) and (3.1), we get

$$H(z) \leqslant 2bH_2(z) + C \leqslant bR|z|^2 + C, \quad \forall z \in \mathbf{R}^{2n}.$$

From the definition of $\hat{H}_K(t,u)$ and the fact that e^{tM} is an orthogonal matrix, we have that

$$\hat{H}_K(t,u) \leqslant 2bH_2\big(e^{tM}u\big) + \frac{1}{2}(JMu,u) + \frac{K}{2}|u|^2 + C \leqslant bR\big|e^{tM}u\big|^2 + \frac{1}{2}\lambda_{\max}|u|^2 + \frac{1}{2}K|u|^2 + C$$
$$= \frac{1}{2}(2bR + \lambda_{\max} + K)|u|^2 + C, \quad \forall u \in \mathbf{R}^{2n}.$$

From the definition of the Fenchel dual we have

$$\hat{H}_K^*(t,u) \geqslant \frac{1}{2(K + 2bR + \lambda_{\max})}|u|^2 - C, \quad \forall u \in \mathbf{R}^{2n}. \tag{3.17}$$

Define

$$A_K(u) = \frac{1}{2}\int_0^1 \big[(K + 2bR + \lambda_{\max})^{-1}|-J\dot{u} + Ku|^2 - (-J\dot{u} + Ku, u)\big]\,dt, \quad \forall u \in \mathcal{W}. \tag{3.18}$$

Using (2.15) and (2.16), we have

$$A_K(u) = \frac{1}{2}\sum_{k\in\mathbf{Z}}\sum_{j=1}^{n}(K + 2bR + \lambda_{\max})^{-1}(K + k\pi)(k\pi - 2bR - \lambda_{\max})u_{kj}^2.$$

Define the subspace V_2 in $\mathcal{W}$ by

$$V_2 = \operatorname{span}\left\{e_{kj}\ \Big|\ -\frac{K}{\pi} < k < \frac{1}{\pi}(2bR + \lambda_{\max}),\ k \in \mathbf{Z},\ \ j = 1,\dots,n\right\}.$$

From the definition of the space V_2 we know that $A_K(u) \geqslant 0$ for $u \in V_2^{\perp}$. From (3.17) and (3.18) we get

$$F_K(u) \geqslant A_K(u) - C \geqslant -C, \quad \forall u \in V_2^{\perp}.$$

Step 3. Since $b := \phi'(+\infty) = \pi(\frac{1}{2R} - s_0)$ and $s_0 \in (0, \frac{1}{4R})$, we know that $\frac{2bR}{\pi} = 1 - 2Rs_0$. Since $a = \pi(\frac{\beta}{R} + \frac{r}{2})$, we know that $\frac{aR}{\beta\pi} = 1 + \frac{rR}{2\beta}$.

Thus we have

$$m - p = n^{\sharp}\left\{k \in \mathbf{Z}\ \Big|\ \frac{2bR + \lambda_{\max}}{\pi} < k < \frac{1}{\pi}\left(\frac{aR}{\beta} + \lambda_{\min}\right)\right\} = n\left(E\left(1 + \frac{rR}{2\beta} + \frac{\lambda_{\min}}{\pi}\right) - E\left(1 - 2Rs_0 + \frac{\lambda_{\max}}{\pi}\right)\right). \qquad \square$$

We note that one can choose $r > 0$ such that $m - p = kn$ for every $k \in \mathbf{Z}_+$.

Proof of Theorem 1.1. By Remark 1.3, it is sufficient to prove Theorem 1.1 for the standard Lagrangian subspace L_0 in $\mathbf{R}^{2n}$. Now, we prove Theorem 1.1 for $L_1 = L_0$ via $\mathbf{Z}_2$-index indirectly under the condition (C).

In fact, we suppose that there are only finitely many (L_0, L)-Lagrangian orbits, without loss of generality, we suppose there are exactly μ pairwise distinct (L_0, L)-Lagrangian orbits.

Step 1. We will prove that we can find $n(\mu + 1)$ pairwise distinct nontrivial critical points of F_K, that is, we can find $n(\mu + 1)$ pairwise distinct nontrivial solutions of the system (2.11).

The even functional F_K satisfies the (PS) condition in view of Lemma 3.4. Now, we choose $r > 0$ such that

$$E\left(1 + \frac{rR}{2\beta} + \frac{\lambda_{\min}}{\pi}\right) - E\left(1 - 2Rs_0 + \frac{\lambda_{\max}}{\pi}\right) = \mu + 1. \tag{3.19}$$

Then from Proposition 3.5, Lemma 3.3 and the choice of r in (3.19), we know that the even functional F_K has at least $m - p = n(\mu + 1)$ pairwise distinct critical points

$$u_{m+1}, u_{m+2}, \dots, u_{m+n(\mu+1)}, -u_{m+1}, -u_{m+2}, \dots, -u_{m+n(\mu+1)},$$

which correspond to the critical values

$$c_{m+1} \leqslant c_{m+2} \leqslant \cdots \leqslant c_{m+n(\mu+1)} < 0$$

defined by

$$c_k = \inf_{\gamma(A)\geqslant k} \sup_{x\in A} F_K(x), \quad k = m+1, m+2, \ldots, m+n(\mu+1),$$

where m and p are the dimensions of the space V_1 and V_2 defined in Proposition 3.5. Since all the above critical values are negative, in view of Proposition 2.2, we know that $u_i, -u_i, i = m+1, m+2, \ldots, m+n(\mu+1)$ are nontrivial solutions of (2.11).

Step 2. we prove that the above $n(\mu+1)$ pairwise distinct nontrivial solutions of the system (2.11) can be transformed to those of the system (2.4), which can be projected on Σ to be $n(\mu+1)$ pairwise distinct (L_0, L)-Lagrangian orbits on Σ.

Set

$$z_i(t) = e^{tM} u_i(t), \quad i = m+1, m+2, \ldots, m+n(\mu+1),$$

then $z_i(t)$, $-z_i(t)$, $i = m+1, m+2, \ldots, m+n(\mu+1)$, are the distinct nontrivial solutions of the system (2.4).

Define the projected orbits by

$$\tilde{z}_i(t) := \rho_i^{-\frac{1}{2}} z_i\left(\frac{\phi'(1)}{\phi'(\rho_i)} t\right), \quad i = m+1, m+2, \ldots, m+n(\mu+1),$$

where $\rho_i = H_2(z_i) > 0$, then $\tilde{z}_i$, $-\tilde{z}_i$, $i = m+1, m+2, \ldots, m+n(\mu+1)$, are nontrivial solutions of the system (2.3) with $\tau_i = \frac{\phi'(\rho_i)}{\phi'(1)}$. That is, $(\tau_i, \tilde{z}_i)$, $(\tau_i, -\tilde{z}_i)$, $i = m+1, m+2, \ldots, m+n(\mu+1)$, are (L_0, L)-Lagrangian orbits on Σ.

Case 1. If $\rho_i \neq \rho_j$, $\forall i \neq j \in \{m+1, m+2, \ldots, m+n(\mu+1)\}$, from the monotone of $\phi'(t)$ we know that $\tau_i = \frac{\phi'(\rho_i)}{\phi'(1)} \neq \frac{\phi'(\rho_j)}{\phi'(1)} = \tau_j$. So $\tilde{z}_i$ and $\tilde{z}_j$ are two distinct (L_0, L)-Lagrangian orbits with different time to end on L.

Case 2. If $\rho_i = \rho_j$, $i \neq j \in \{m+1, m+2, \ldots, m+n(\mu+1)\}$, then z_i and z_j are two distinct solutions of (2.4) with the same energy. So $\tilde{z}_i$ and $\tilde{z}_j$ are two distinct (L_0, L)-Lagrangian orbits on Σ with the same time to end on L.

From above discussion, we know that no matter what case happens, there exist at least $n(\mu+1) > \mu$ pairwise distinct Lagrangian orbits on Σ, which contradicts our assumption that there are exactly μ pairwise distinct Lagrangian orbits on Σ. The proof is completed. □

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